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Crossing Symmetry in S-Matrix Theory*

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A general method is given to calculate the crossing relations for arbitrary processes which is based on the transformation properties of the amplitudes and analytic continuation only. The helicity amplitudes as well as the so called R and M amplitudes are considered and some simple examples are worked out.

I

ONE of the most important practical elements of dynamical calculations which makes use of the analytical properties of the scattering amplitudes is the crossing symmetry.[1] It relates various amplitudes, for example helicity amplitudes, in one channel to those in other channels, or more generally to other processes in which one or more or all incoming and outgoing particles have been interchanged. Roughly speaking, the processes in crossed channels provide forces for the process in the original channel. For example, forces responsible for the binding of two particles may be attributed to the exchange of other particles in crossed channels. As is well known, the crossing relations generalize Pauli exchange principle.

The crossing matrices for simple cases such as $\pi-N$, $N-N$ processes are well known.[2,3] It is desirable to calculate explicitly the crossing matrices for higher spin processes, in particular in connection with the new resonances in strong interactions. It is the purpose of this work to show how to calculate the crossing relations within the framework of the so called S-matrix theory, based solely on the transformation properties of the amplitudes and analytic continuation and to present a method valid for arbitrary processes with arbitrary spin values.

The crossing relations for arbitrary values of isotopic spin have been already worked out within the same approach.[4] The spin case discussed below is more complicated because spin transformations are coupled to those of momenta.

We use two-component spinors throughout which brings considerable simplification. Some useful relations used in the following calculations are summarized in the Appendix.

II

We consider for simplicity of writing a two-body reaction process $a+b\rightarrow c+d$, although the formulas are such that they are immediately generalized to arbitrary processes. The amplitudes R $(R=S-I)$, are invariant under the inhomogeneous Lorentz transformations and satisfy for massive particles the following equation[5,6]:

$$\begin{aligned}R_{m_3m_4m_1'm_2'}(k_3k_4,k_1k_2)&=\mathfrak{D}_{m_3}{}^{(S_3)n_3}(A')\mathfrak{D}_{m_4}{}^{(S_4)n_4}(A')\\&\quad\times\mathfrak{D}_{m_1'}{}^{(S_1)*n_1'}(A')\mathfrak{D}_{m_2'}{}^{(S_2)*n_2'}(A')\\&\quad\times R_{n_3n_4n_1'n_2'}(\Lambda^{-1}k_3\Lambda^{-1}k_4,\Lambda^{-1}k_1\Lambda^{-1}k_2).\quad(1)\end{aligned}$$

Here, m_3 and m_4 are the spin indices (the third component of the spin) of the outgoing particles k_3 and k_4 of spins S_3 and S_4 $(-S_3<m_3<S_3$ etc.) and m_1' and m_2' are those of the incoming particles k_1 and k_2. We have taken, by convention, the incoming particles and outgoing antiparticles to transform with $\mathfrak{D}^*$ and the outgoing particles and incoming antiparticles with $\mathfrak{D}$, the irreducible representations of the rotation group associated with the spin in the rest frame; Λ is a homogeneous Lorentz transformation. The 2 by 2 unitary matrix

$$A'=B^{-1}{}_{k\leftarrow p}AB_{q\leftarrow p}\quad(2)$$

corresponds to a special Lorentz transformation, a rotation, which takes the momentum in the rest frame, p, first into $q=\Lambda^{-1}k$, then into k, the physical momentum and then back to p; A' is an element of the so-called little group of the vector p. With $p=(m,0,0,0)$ we have

$$B_{k\leftarrow p}=(k^\mu\sigma_\mu/m)^{1/2}U,\quad(3)$$

where U is any unitary operator corresponding to a rotation. If one of the particles has zero mass, the corresponding $\mathfrak{D}$ term in Eq. (1) is replaced by a factor of $\exp(iS\varphi)$, where S is the helicity of the particle with two states $\pm S$.

For general processes simply more $\mathfrak{D}$ factors are added in Eq. (1).

The spin of the particle is associated with the rotation group in the rest frame. The choice $U=1$ in Eq. (3) corresponds to measuring the third component of the spin with respect to a fixed z axis. We denote the corresponding amplitudes the R *amplitudes*. If we make a rotation to bring the z axis in the direction of the motion

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[1] For a general discussion of the role of crossing symmetry, see G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962).

[2] For π-N problem, see, for example, S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[3] For N-N problem, see M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

[4] A. O. Barut and B. C. Unal, Nuovo Cimento (to be published).

[5] A. O. Barut, Phys. Rev. **127**, 3 (1962).

[6] A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. (to be published).

of the particle by choosing

$$U=\exp(\tfrac{1}{2}i\sigma_3\varphi)\exp(\tfrac{1}{2}i\sigma_2\theta)\exp(\tfrac{1}{2}i\sigma_3\varphi), \quad (4)$$

where φ and θ are the angles of the spatial part of k, we obtain the H (*helicity*) *amplitudes*.

III

The crossing relations for these R or H amplitudes are obtained as follows. Suppose we interchange one outgoing particle with one incoming one, m_3 and m_1'. We obtain from the original amplitude $R_{m_3m_4m_1'm_2'}(k_3k_4,k_1k_2)$ the amplitude

$$\bar{R}_{m_1'm_4m_3m_2'}(-k_1k_4,\,-k_3k_2). \quad (5)$$

For the R functions, transforming with $\mathfrak{D}^{(S)}$, the primed indices are equivalent to upper indices. Therefore, using the lowering and raising spinor operators (see Appendix)

$$\mathfrak{D}_{mn}{}^{(S)}(C)=(-1)^{S-m}\delta_{m-n};\qquad \mathfrak{D}^{(S)}(C^{-1})^{mn}=(-1)^{S+m}\delta^{m,-n}, \quad (6)$$

we can write the amplitude (5) in the form

$$\mathfrak{D}^{(S_1)m_1n_1}(C^{-1})\mathfrak{D}^{(S_3)}(C)_{m_3n_3}\bar{R}_{n_1m_4n_3'm_2'}(-k_1k_4,\,-k_3k_2) \quad (7)$$

or

$$(-1)^{S_1+S_3+m_1-m_3}\bar{R}_{-m_1m_4-m_3m_2'}(-k_1k_4;\,-k_3k_2). \quad (8)$$

The R function in (7) refers to an amplitude for a process in which particles k_1, k_4 are outgoing and k_3, k_2 are incoming. The crossing relation is then

$$R_{m_3m_4m_1'm_2'}(k_3k_4,\,k_1k_2)=\mathfrak{D}^{(S_1)}(C^{-1})_{m_1'}{}^{n_1}\mathfrak{D}^{(S_3)}(C)_{m_3}{}^{n_3'}\times\bar{R}_{n_1m_4m_3'm_2'}(-k_1k_4,\,-k_3k_2). \quad (9)$$

In these equations all four-vectors k_i are the physical energy-momentum vectors, and $\bar{R}$ refers to the amplitude in crossed channel. Equation (8) shows that actually in the crossed process the spin components of the exchanged particles are the negatives of the original ones. The same type of equation holds also for the isospin,[4] and the exchange particles become then "antiparticles." Indeed, using the relation

$$\mathfrak{D}_\alpha{}^{(S)\,\beta}=(-1)^{\alpha-\beta}\mathfrak{D}^{S*}{}^{-\alpha}{}_{-\beta}, \quad (10)$$

one can easily show that both sides of Eq. (9) have the same transformation property given by Eq. (1).

Similar equations are obtained in the general case and for the exchange of other particles.

The crossing matrices themselves refer to the scalar amplitudes. If we express both sides of Eq. (9) in terms of spinor basis functions multiplied by scalar coefficients in the form

$$A_i(s,t,u)Y_i(k_3k_4,k_1k_2)\quad\text{and}\quad \bar{A}_i(u,t,s)\bar{Y}_i(-k_1k_4,\,-k_3k_2),$$

respectively, and compare the basis function, we obtain the crossing matrix β satisfying

$$\textstyle\sum_{i'}\beta_{ii'}Y_{i'}(k_3k_4,k_1k_2)=\bar{Y}_i(-k_1k_4;\,-k_3k_2). \quad (11)$$

Now the construction of the spinor basis functions is accomplished most conveniently in terms of the M amplitudes to be defined below and is discussed in detail in reference 4. We therefore turn to a discussion of the crossing properties of the M amplitudes and the scalar amplitudes.

IV

The M amplitudes arise from a natural simplification of Eq. (1). Because A' given by Eq. (2) is unitary we can write

$$\mathfrak{D}^{(S)}(B^{-1}AB)=\mathfrak{D}^{(0S)}(B^{-1}AB)=\mathfrak{D}^{(0S)}(B^{-1})\mathfrak{D}^{(0S)}(A)\mathfrak{D}^{(0S)}(B),$$

where $\mathfrak{D}^{(SS')}$ are the irreducible representations of the homogeneous Lorentz group. Note that B and A are unimodular but not unitary. If we now define

$$M_{m_3m_4m_1'm_2'}=\mathfrak{D}_{m_3}{}^{(0S_3)n_3}(B)\mathfrak{D}_{m_4}{}^{(0S_4)n_4}(B)\times\mathfrak{D}_{m_1'}{}^{(0S_1)n'}(B^*)\mathfrak{D}_{m_2'}{}^{(0S_2)n_2'}(B^*)R_{n_3n_4n_1'n_2'}, \quad (12)$$

then we see from (1) that M functions transform according to irreducible representations of the homogeneous Lorentz groups in the form

$$M_{m_3m_4m_1'm_2'}(K)=\mathfrak{D}_{m_1'}{}^{(S_10)*n_1'}(A)\mathfrak{D}_{m_2'}{}^{(S_20)*n_2'}(A)\mathfrak{D}_{m_3}{}^{(S_30)n_3}(A)\times\mathfrak{D}_{m_4}{}^{(S_40)n_4}(A)M_{n_3n_4n_1'n_2'}(\Lambda^{-1}K). \quad (13)$$

To discuss the crossing symmetry of the M amplitudes we have to relate $\mathfrak{D}^{(S0)}(A)$ to $\mathfrak{D}^{(S0)}(A)^*$, or change primed indices into unprimed and vice versa. Because now these two represntations are inequivalent there is no simple relation corresponding to Eq. (10). These two representations are related by k-dependent matrices as follows: For 2 by 2 matrices we have for any four-vector k the relation

$$C(k\cdot\bar{\sigma}/m)A[(\Lambda^{-1}k)\cdot\sigma/m]C^{-1}=A^* \quad (14)$$

which is a consequence of the relation $C^{-1}A^TC=A^{-1}\det A$ valid for any 2 by 2 matrix and $Ak\cdot\sigma A^\dagger=\Lambda k\cdot\sigma$ which relates the Lorentz group Λ to the unimodular group A. From the group property of 2×2 matrices the generalization of (10) is

$$G^S(k)_{\alpha'}{}^{\alpha}\mathfrak{D}^{(S0)}(A)_\alpha{}^\beta G^{-1S}(\Lambda^{-1}k)_\beta{}^{\beta'}=\mathfrak{D}^{(S0)}(A)_{\alpha'}{}^{\beta'}, \quad (15)$$

where

$$G^S(k)=\mathfrak{D}^{(S0)}[C(k\cdot\bar{\sigma}/m)]. \quad (16)$$

Consider again the interchange of particles k_1 and k_3, for example. The steps are analogous to Eqs. (5) to (9); we have to use the operators G and G^{-1} instead of $\mathfrak{D}(C)$ and $\mathfrak{D}(C)^{-1}$. We obtain the crossing relation

$$M_{m_3m_4m_1'm_2'}(k_3k_4,k_1k_2)=G(k_1)_{m_1'}{}^{n_1}G^{-1}(k_3)_{m_3}{}^{n_3'}\times\bar{M}_{n_1m_4n_3'm_2'}(-k_1k_4,\,-k_3k_2). \quad (17)$$

Again it can be verified, using (15) and its inverse, that both sides of (17) have the correct transformation property given by Eq. (1).

To obtain the relation between the scalar amplitudes we have to expand both M functions in (17) into spinor basis functions, do the G operations on the right and then compare the scalar coefficients as in Eq. (11). Let us consider a simple example, the scattering of one spin-zero particle and one spin-$\frac{1}{2}$ particle. Let k_1 and k_3 be the particles with spin $\frac{1}{2}$. We have

$$M_{m_3 m_1'} = A_i Y_i(k_3 k_4;\, k_1 k_2),$$

where a basis with definite signature under P and T is the following[7]:

$$\begin{aligned}
Y_1 &= \left(\frac{k_1}{m_1}+\frac{k_3}{m_3}\right)\cdot\sigma, \quad \eta_p=+1, \quad \eta_T=+1,\\
Y_2 &= \left(\frac{k_1}{m_1}-\frac{k_3}{m_3}\right)\cdot\sigma, \quad \eta_p=-1, \quad \eta_T=-1,\\
Y_3 &= n\cdot\sigma+(k_3\cdot\sigma/m_3)n\cdot\tilde{\sigma}(k_1\cdot\sigma/m_1),\\
&\qquad\qquad \eta_p=+1, \quad \eta_T=+1,\\
Y_4 &= n\cdot\sigma-(k_3\cdot\sigma/m_3)n\cdot\tilde{\sigma}(k_1\cdot\sigma/m_1),\\
&\qquad\qquad \eta_p=-1, \quad \eta_T=+1,
\end{aligned} \tag{18}$$

where

$$n=\left(\frac{k_2}{m_2}+\frac{k_4}{m_4}\right).$$

Now according to (17) we have to write the bases $Y_i(-k_1k_4, -k_3k_2)$, i.e., k_1 and k_3 interchanged in Eq. (18), and then operate with the G's. In matrix rotation

$$A_iY_i(k_3k_4,k_1k_2) = \frac{k_3\cdot\sigma}{m_3}C^{-1}\bar{A}_{i'}\bar{Y}_{i'}{}^T(-k_1k_4;\,-k_3k_2)\left(C\frac{k_1\cdot\tilde{\sigma}}{m_1}\right)^T$$

which leads to the crossing matrix $A_i=\beta_{ii'}\bar{A}_{i'}$, where

$$\beta_{ii'}=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1\end{bmatrix}. \tag{19}$$

In terms of the usual invariants s, t, u we have

$$A_i(s,t,u)=\beta_{ii'}\bar{A}_{i'}(u,t,s). \tag{20}$$

The amplitudes A_1 and A_3 are the same as the usual A and B amplitudes in π-N scattering.[6]

Next we consider the scattering of two spin-$\frac{1}{2}$ particles and a basis which is a direct product of two bases of the form (18) Y_iZ_j. [Z_i are given by (18) with k_1 replaced by k_2 and k_3 replaced by k_4.] The crossing matrix between the 16 amplitudes A_{ij} is now

$$\begin{aligned} A_{ij}(s,t,u) &= \beta_{ii'}\delta_{jj'}\bar{A}_{i'j'}(u,t,s)\\ &= \beta_{ii'}\bar{A}_{i'j'}(u,t,s). \end{aligned} \tag{21}$$

[7] H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

The basis used in reference 3 for $N-N$ scattering is not the product basis Y_iZ_j, but the so-called β-decay basis in four-component form. We have written the β-decay basis in terms of the two-component spinors and have verified that Eq. (17) leads exactly to the crossing matrix given in reference 3; the corresponding helicity amplitudes for this problem also agree with the choice of U as given by Eq. (4).

V

Finally, we should like to remark that the need for crossing matrixes arises because one uses different basis functions for the original and the crossed channels, denoted by Y_i and $\bar{Y}_i$ in the previous section. Correspondingly, the arguments of the scalar amplitudes A_i and $\bar{A}_i$ are different. Because, however, these two amplitudes, and others in the remaining crossed channels, are connected to each other by analytic continuation, one can also simply use the analytic continuation of the original basis functions. These are obtained by changing the signs of the momenta corresponding to the particular crossed channel considered. For example, the functions $Y_i(-k_3, k_4; -k_1, k_2)$ (no permutation, just change of signs) describe the process $2+\bar{3}\to\bar{1}+4$, if the functions $Y_i(k_3,k_4;k_1,k_2)$ describe the process $1+2\to 3+4$, and similarly for the other channels. If this method is used, the scalar amplitudes $A_i(s,t,u)$ are the same functions for all channels, but the physical domain of the invariants s, t, u are now those of the crossed channels. This is, indeed, the generalization of what happens in the spinless case and one does not need any crossing matrices. Although this approach does not seem to have been used in literature, it appears that it would simplify practical problems considerably.

I should like to thank H. P. Stapp, D. N. Williams, and I. Muzinich for some helpful correspondence and discussion.

APPENDIX

We review here briefly some properties of spinors used in the text.

Spinors transforming according to A, A^{T-1}, A^*, and $A^{\dagger-1}$ are written with lower unprimed, upper unprimed, lower primed, and upper primed indices. If A is unitary, upper unprimed and lower primed indices are the same, and vice versa. The lowering and rising matrices are $C_{\alpha\beta}$ and $C^{-1\alpha\beta}$, respectively, where

$$C=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\equiv -C^{-1}\equiv -C^T=-C^\dagger.$$

In the generalizations of these statements to higher spinors we replace A by $D^{(0S)}(A)$, C by $D^{(0S)}(C)$ etc.

We denote the Pauli matrices by $\sigma_\mu=(I,\boldsymbol{\sigma})$. Then $\sigma^\mu=(I,-\sigma)=\tilde{\sigma}_\mu$. We write frequently $k\cdot\sigma=k^\mu\sigma_\mu$.

One gets

$$k\cdot\sigma k\cdot\bar{\sigma}=m^2,$$

$(k\cdot\sigma)^{1/2}$ is the Hermitian square root of $k\cdot\sigma$.

The relation between the unimodular matrices and the restricted Lorentz transformations is given by

$$A\sigma_\mu A^\dagger=\Lambda_\mu{}^\nu\sigma_\nu,$$

or

$$\Lambda^\mu{}_\nu(\pm A)=\tfrac{1}{2}\,\mathrm{Tr}(\bar{\sigma}^\mu A\sigma_\nu A^\dagger).$$

We have also

$$\tfrac{1}{2}\,\mathrm{Tr}(\bar{\sigma}^\mu\bar{\sigma}_\nu{}^T)=g_\nu{}^\mu,$$

$$\sigma_\mu{}^T=C\bar{\sigma}_\mu C^{-1}\quad\text{or}\quad\sigma_\mu=C\bar{\sigma}_\mu{}^TC^{-1}.$$

For any 2 by 2 matrix M the relation $CM^TC^{-1}=M^{-1}\det M$ is an identity.

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Plasmons, Gauge Invariance, and Mass

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Schwinger has pointed out that the Yang-Mills vector boson implied by associating a generalized gauge transformation with a conservation law (of baryonic charge, for instance) does not necessarily have zero mass, if a certain criterion on the vacuum fluctuations of the generalized current is satisfied. We show that the theory of plasma oscillations is a simple nonrelativistic example exhibiting all of the features of Schwinger's idea. It is also shown that Schwinger's criterion that the vector field $m\neq 0$ implies that the matter spectrum before including the Yang-Mills interaction contains $m=0$, but that the example of superconductivity illustrates that the physical spectrum need not. Some comments on the relationship between these ideas and the zero-mass difficulty in theories with broken symmetries are given.

RECENTLY, Schwinger[1] has given an argument strongly suggesting that associating a gauge transformation with a local conservation law does not necessarily require the existence of a zero-mass vector boson. For instance, it had previously seemed impossible to describe the conservation of baryons in such a manner because of the absence of a zero-mass boson and of the accompanying long-range forces.[2] The problem of the mass of the bosons represents the major stumbling block in Sakurai's attempt to treat the dynamics of strongly interacting particles in terms of the Yang-Mills gauge fields which seem to be required to accompany the known conserved currents of baryon number and hypercharge.[3] (We use the term "Yang-Mills" in Sakurai's sense, to denote any generalized gauge field accompanying a local conservation law.)

The purpose of this article is to point out that the familiar plasmon theory of the free-electron gas exemplifies Schwinger's theory in a very straightforward manner. In the plasma, transverse electromagnetic waves do not propagate below the "plasma frequency," which is usually thought of as the frequency of long-wavelength longitudinal oscillation of the electron gas. At and above this frequency, three modes exist, in close analogy (except for problems of Galilean invariance implied by the inequivalent dispersion of longitudinal and transverse modes) with the massive vector boson mentioned by Schwinger. The plasma frequency is equivalent to the mass, while the finite density of electrons leading to divergent "vacuum" current fluctuations resembles the strong renormalized coupling of Schwinger's theory. In spite of the absence of low-frequency photons, gauge invariance and particle conservation are clearly satisfied in the plasma.

In fact, one can draw a direct parallel between the dielectric constant treatment of plasmon theory[4] and Schwinger's argument. Schwinger comments that the commutation relations for the gauge field A give us one sum rule for the vacuum fluctuations of A, while those for the matter field give a completely independent value for the fluctuations of matter current j. Since j is the source for A and the two are connected by field equations, the two sum rules are normally incompatible unless there is a contribution to the A rule from a free, homogeneous, weakly interacting, massless solution of the field equations. If, however, the source term is large enough, there can be no such contribution and the massless solutions cannot exist.

The usual theory of the plasmon does not treat the electromagnetic field quantum-mechanically or discuss vacuum fluctuations; yet there is a close relationship between the two arguments, and we, therefore, show that the quantum nature of the gauge field is irrelevant. Our argument is as follows:

The equation for the electromagnetic field is

$$p^2A_\mu=(k^2-\omega^2)A_\mu(\mathbf{k},\omega)=4\pi j_\mu(\mathbf{k},\omega).$$

[1] J. Schwinger, Phys. Rev. **125**, 397 (1962).
[2] T. D. Lee and C. N. Yang, Phys. Rev. **98**, 1501 (1955).
[3] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1961).
[4] P. Nozières and D. Pines, Phys. Rev. **109**, 741 (1958).